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Electrostatics of the Point Dipole and Higher Multipoles

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It is shown that a distribution theory form for the electrostatics of point multipoles can be constructed, which reduces to the usual theory away from the position of the multipole, but in which the total electromagnetic self-energy is zero and the self-force density is zero.

1. INTRODUCTION

Dirac's paper (Dirac, 1938) on the classical electrodynamics of charged point particles is a key landmark in the subject. If it did not solve all the problems—indeed, it created many new ones—everything written after it has had Dirac's paper as a significant point of reference.

Dirac set out to derive the equation of motion of the charged point particle, the equation that is now known as the Lorentz–Dirac equation, by working with Maxwell's equations and considerations of energy-momentum conservation. He calculated the electromagnetic energy-momentum flux over the surface of a tube in space-time surrounding the world line of the point charge. For a tube of finite length and nonzero width this is a perfectly finite, calculable quantity. He then implemented energy-momentum conservation in a rather formal way using an argument in which he drew back from giving *a priori* status to any special form for the point particle's four-momentum (in the end, as a matter only of simplicity, he chose mv), and in which he delayed the appearance of divergent integrals that would have resulted if he had tried to calculate the electromagnetic energy at the ends of the tube. But in order to deduce an equation of motion, he had at last to take a limit as the width of the tube went to zero. This process produced divergent terms that he canceled with correspondingly infinite mass renormalization terms.

Dirac's procedure may be regarded as an attempt to satisfy energy-momentum conservation in the form

$$\int dS\cdot T=\int dS\cdot(K+\theta)=0 \tag{1}$$

where T is the total (stress-) energy (-momentum) tensor, K is the particle's kinematic energy tensor, and θ is the electromagnetic energy tensor; the integral is taken over the surface of a four-dimensional region in space-time. In trying to formulate the electrodynamics of point particles one has the problem of specifying K, and the problem of divergences when θ, treated as a tensor-valued function, is integrated over a region cut by the world line.

The mass renormalization process, in which an ad hoc divergence is introduced in the integral $\int dS\cdot K$ to cancel the apparently unavoidable one in $\int dS\cdot\theta$, can only be regarded as a temporary patch on the theory. One cannot really integrate a function with a singularity r^{-4} as if it were finite and summable, but the mathematics which could provide a cure was not created until the late 1940s (Schwartz, 1973; Gelfand and Shilov, 1964). If one reformulates the problem using distributions rather than functions, one gets a natural generalization of the integral concept which agrees with the old concept for sufficiently regular functions, but gives well-defined finite values for the "divergent" integrals of electrodynamics. There should be no hesitation in accepting the distribution idea for the energy tensor since it is readily used in Maxwell's equations as the source for point particles: using anything other than distributions for θ makes the mathematics manifestly inconsistent.

In the present note, in an attempt to overcome divergence difficulties in their simplest possible context, we consider only isolated and static point multipoles. If the physical situation is static—or in uniform motion, but we will generally take the rest frame's point of view—we are justified in supposing that the particle's kinematic four-momentum and the electromagnetic four-momentum are conserved separately. The differential form of (1) is

$$\partial\cdot T=\partial\cdot(K+\theta)=0 \tag{2}$$

but in the static case we expect it to be realized in a more particular way:

$$\partial\cdot K=0 \tag{3}$$

$$\partial\cdot\theta=0 \tag{4}$$

If the kinematic energy tensor K has its simplest form,

$$K = \int d\tau\, mvv\delta(x - z)$$

then

$$\partial \cdot K = \int d\tau \frac{d}{d\tau}(mv)\delta(x - z)$$

This accounts for the designation "force density" for $f = -\partial \cdot \theta$ even in the nonstatic case. Whether K has its simplest form or not, we think of the condition (4) as characterizing zero force on the point particle, and we can try to solve (4) without making any special assumptions about K. There is no interchange of momentum between the electromagnetic field and the particle when (4) is satisfied.

There are two important reasons for solving (4). First, it would provide genuine solutions of the equations expressing four-momentum conservation in the case of charged point particles interacting with the electromagnetic field. Such genuine solutions, if they exist at all in the literature, are sufficiently rare to merit attention. Secondly, the difficulty in solving (2) in the general case of arbitrary motion is so great that one needs the clues offered by the static case. We expect the most singular terms in the energy tensor for the general case to be just those that arise in the static case; radiation will be described by less singular terms.

In the later sections we will discuss higher multipoles, but it may be useful to summarize here the electrostatics of the single point charge since this case has already been dealt with in a mathematically satisfactory way (Section VII in Rowe, 1978), and it can serve as a model for the other multipoles.

We will focus attention on two aspects of the electrostatics of the point particle in which the singularities in the classical treatment create acute problems: the self-energy and the self-force. For a stationary point charge e at $\mathbf{0}$, the fields, defined as functions for $\mathbf{x} \neq \mathbf{0}$, are static:

$$\mathbf{E}(\mathbf{x}) = \frac{e\mathbf{x}}{r^3}, \qquad \mathbf{B}(\mathbf{x}) = \mathbf{0} \qquad (r > 0) \tag{5}$$

Off the world line the electromagnetic energy density may be written

$$\theta^{00} = \frac{\mathbf{E}^2}{8\pi} = \frac{e^2}{8\pi r^4} = \frac{e^2}{16\pi}\nabla^2\frac{1}{r^2} \qquad (r > 0) \tag{6}$$

and the momentum flux tensor (minus Maxwell's stress tensor) may be written as the following dyadic in three-space:

$$\theta^{ss} = \frac{I}{8\pi}\mathbf{E}^2 - \frac{\mathbf{EE}}{4\pi} = \frac{e^2}{16\pi}\nabla^2\frac{\mathbf{xx}}{r^4} \qquad (r>0) \tag{7}$$

In (7) I stands for the unit dyadic (for example, $I\cdot\mathbf{E}=\mathbf{E}\cdot I=\mathbf{E}$) and the superscript ss refers simply to the space space sector of the four-dimensional tensor θ. The Poynting vector $\mathbf{E}\times\mathbf{B}/4\pi=\boldsymbol{\theta}^{0s}=\boldsymbol{\theta}^{s0}$ is zero for $r>0$.

The integral of θ^{00}, regarded as a function, over a region including the point $\mathbf{0}$, does not exist, not even if one excludes the origin and takes a limit. The force density, $\mathbf{f}(\mathbf{x})=-\nabla\cdot\theta^{ss}=\rho(\mathbf{x})\mathbf{E}(\mathbf{x})$ for $r>0$, vanishes at all points away from the origin but has no value or meaning at the origin. In elementary treatments of point electrostatics the self-force is simply ignored, a procedure which is consistent with using the value zero expected from a rigorous treatment. The problem we have is to understand the energy density θ^{00} in a new way so its integral exists, and to understand θ^{ss} in a new way so its divergence exists and gives us information about the self-force.

If we interpret the derivative forms of equations (6) and (7) in a distribution theory sense, that is, as the distribution theory Laplacian acting on the regular integrable functions $1/r^2$ and $\mathbf{xx}/r^4$, we get a solution for our electrostatics problems and sensible, consistent expressions for the energy and force density. We therefore consider the theory in which θ^{00} and θ^{ss} are distributions defined by

$$\theta^{00} = \frac{e^2}{16\pi}\nabla^2\frac{1}{r^2} \tag{8}$$

$$\theta^{ss} = \frac{e^2}{16\pi}\nabla^2\frac{\mathbf{xx}}{r^4} \tag{9}$$

The question why precisely (8) and (9) are the definitions, and why there are no extra terms on the right-hand side, δ functions and their derivatives, is important. It is the uniqueness problem for θ^{00} and θ^{ss}, which are specified by (6) and (7) only in the region $r>0$. The answer requires a consideration of the general, nonstatic case and is the main subject of the paper referred to above (Rowe, 1978). In the present note we do not attempt the uniqueness question but content ourselves with finding possible forms for the energy tensor that yield finite results consistent with the zero-force condition for a static situation.

From (8) we can show that the total electromagnetic energy of the point charge is zero, not infinite. The total energy, if we use the new distribution theory definitions for θ^{00} and for the integral, and take a limit of the total energy within a sphere of radius $r = E$, is

$$\begin{aligned}
\int d\mathbf{x}\,\theta^{00} &\equiv \lim_{E\to\infty}\int d\mathbf{x}\,\theta^{00}\vartheta(E-r)\\
&\equiv \lim_{E\to\infty}\left(\theta^{00},\vartheta(E-r)\right)\\
&= -\frac{e^2}{16\pi}\lim_{E\to\infty}\int r^2 dr\,d\Omega\,\nabla\frac{1}{r^2}\cdot\nabla\vartheta(E-r)\\
&= -\frac{e^2}{8\pi}\lim_{E\to\infty}\int r^2\,dr\,d\Omega\,\frac{1}{r^3}\delta(r-E)\\
&= -\lim_{E\to\infty}\frac{e^2}{2E}=0
\end{aligned} \tag{10}$$

The notation of (Gelfand and Shilov, 1964) has been used. Notice that the energy in a finite sphere is negative. Despite the fact that (8) is equivalent, off the world line, to a positive definite function, the distribution is not positive definite. Distributions which are positive definite must be regular (Schwartz, 1973), and since θ^{00} is not regular there is no contradiction.

From (9) we can show that the self-force, indeed the self-force density, is zero. The density is

$$\mathbf{f} = -\nabla\cdot\theta^{ss} = -\frac{e^2}{16\pi}\nabla^2\nabla\cdot\left(\frac{\mathbf{xx}}{r^4}\right) \tag{11}$$

It is clear that

$$\nabla\cdot\left(\frac{\mathbf{xx}}{r^4}\right)=0 \qquad \text{for } r>0 \tag{12}$$

and in general, with any test function ϕ,

$$\left(\nabla\cdot\frac{\mathbf{xx}}{r^4},\phi\right)=-\int d\mathbf{x}\,\nabla\phi\cdot\frac{\mathbf{xx}}{r^4}$$

$$=-\lim_{\epsilon\to 0}\int d\mathbf{x}\,\nabla\phi\cdot\frac{\mathbf{xx}}{r^4}\vartheta(r-\epsilon)$$

$$=\lim_{\epsilon\to 0}\int d\mathbf{x}\,\phi\frac{\mathbf{xx}}{r^4}\cdot\nabla\vartheta(r-\epsilon)$$

$$=\lim_{\epsilon\to 0}\int r^2\,dr\,d\Omega\,\phi\frac{\mathbf{x}}{r^3}\delta(r-\epsilon)=\mathbf{0} \tag{13}$$

Consequently the formula (12) is true *without* the restriction $r>0$.

In turning to the study of classical point particle electrodynamics Dirac wanted to understand that subject better not only for its own sake but also to get clues, ideas, reformulations which might assist in overcoming the divergence problems in quantum electrodynamics. The spectacularly accurate numerical predictions of the quantum field theory of the spinning electron, with its charge and magnetic moment in interaction with the quantized electromagnetic field, have overshadowed the theory's rather shaky foundations, its divergences, its infinite renormalizations and its probably nonconvergent perturbation series. The same motivation, to apply increased understanding of the classical case to quantum electrodynamics, still animates us today.

Immediately after Dirac's pioneering work, attempts were made (Bhabha and Corben, 1941) to treat similarly a more realistic classical model of the point electron: a particle with both charge and a general (magnetic and electric) dipole moment. The calculations were much much longer, the singularities even worse, the arbitrariness on the kinematical side (for example, parameters describing moments of inertia) greater, and the resulting equation of motion so complicated that it defied physical understanding [relatively simple through it is, the Lorentz–Dirac equation for the plain point charge has only recently been understood physically (Teitelboim, 1970)]. Even today the theory of the classical dipole particle is largely phenomenological (Teitelboim, Villarroel, and van Weert, 1980).

In Sections 2 and 3 we construct the very first stage of a new beginning in the classical theory of higher point multipoles. In Section 2 we construct an electromagnetic energy tensor which allows a consistent static theory for a point electric dipole (a simple transformation could be used to convert this to the static theory of a magnetic dipole). In Section 3 we sketch the extension to the point quadrupole and higher multipoles.

2. ELECTROSTATICS OF THE POINT DIPOLE

For a time-independent electric dipole **p** at **0** the electric field is the distribution

$$\mathbf{E} = -\nabla \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} = \nabla(\mathbf{p}\cdot\nabla)\frac{1}{r} \tag{14}$$

In the region $r > 0$ it is equivalent to the vector function

$$\mathbf{E}(\mathbf{x}) = \frac{-\mathbf{p}r^2 + 3\mathbf{x}(\mathbf{p}\cdot\mathbf{x})}{r^5} \qquad (r>0) \tag{15}$$

The distinction between (14) and (15) is most evident when one takes divergences. The divergence of (14) is $-4\pi(\mathbf{p}\cdot\nabla)\delta(\mathbf{x})$, which is 4π times the charge density for the dipole, whereas the divergence of (15) is zero in the region $r > 0$, and does not exist at $r = 0$.

For the region $r > 0$ the energy density may be written as a function in the form

$$\begin{aligned}\theta^{00} &= \frac{\mathbf{p}^2 r^2 + 3(\mathbf{p}\cdot\mathbf{x})^2}{8\pi r^8} \qquad (r>0)\\ &= -\frac{1}{64\pi}(\nabla^2)^2\left[\frac{(\mathbf{p}\cdot\mathbf{x})^2 - \mathbf{p}^2 r^2}{r^4}\right] \qquad (r>0)\end{aligned} \tag{16}$$

and similarly, the momentum flux tensor may be expressed

$$\begin{aligned}\theta^{ss} &= \frac{1}{4\pi}\left\{\frac{I}{2}\left[\frac{3(\mathbf{p}\cdot\mathbf{x})^2 + \mathbf{p}^2 r^2}{r^8}\right] - \frac{\mathbf{pp}}{r^6}\right.\\ &\qquad \left. - \frac{9\mathbf{xx}(\mathbf{p}\cdot\mathbf{x})^2}{r^{10}} + \frac{3(\mathbf{p}\cdot\mathbf{x})(\mathbf{xp}+\mathbf{px})}{r^8}\right\} \qquad (r>0)\\ &= -\frac{1}{64\pi}(\nabla^2)^2\left[\frac{\mathbf{xx}\left[(\mathbf{p}\cdot\mathbf{x})^2 - \mathbf{p}^2 r^2\right]}{r^6}\right] \qquad (r>0)\end{aligned} \tag{17}$$

The derivative forms in (16) and (17) require a few pages of straightforward vector analysis to verify. Doing (17) is sufficient because its trace equals (16).

The meaning of the derivative forms may be extended to give well-defined distributions which are equal to the functional forms of θ^{00} and θ^{ss} off the world line. We look at the properties of the theory defined by this generalized interpretation of the derivatives. As yet we have no argument why these definitions should not be supplemented by further terms concentrated on the world line, so the status of this theory of the dipole is different from that of the point charge whose energy density and momentum flux are necessarily given by (8) and (9) if energy-momentum is to be conserved. But the generalized interpretations of (16) and (17) are encouraging because they lead, as (8) and (9) do, to zero energy and zero force, results we intuitively expect.

We find zero total electromagnetic energy by a calculation like that in (10):

$$\begin{aligned}\int d\mathbf{x}\,\theta^{00} &= \lim_{E\to\infty}\left(\theta^{00},\vartheta(E-r)\right)\\ &\equiv \frac{1}{64\pi}\lim_{E\to\infty}\int d\mathbf{x}\,\nabla\vartheta(E-r)\cdot\nabla\nabla^2\left[\frac{(\mathbf{p}\cdot\mathbf{x})^2-\mathbf{p}^2r^2}{r^4}\right]\\ &=0 \end{aligned} \tag{18}$$

The second result, that the force density vanishes, $\mathbf{f}=-\nabla\cdot\theta^{ss}=\mathbf{0}$, follows from

$$\nabla\cdot\left\{\frac{\mathbf{x}\mathbf{x}\left[(\mathbf{p}\cdot\mathbf{x})^2-\mathbf{p}^2r^2\right]}{r^6}\right\}=\mathbf{0} \tag{19}$$

this is true for $r>0$ by standard vector analysis, and true as a distribution theory equation by an argument almost identical to that in (13).

Consequently, the point dipole at rest is a consistent physical system if its energy density and momentum flux are given by the generalized forms of (16) and (17). Its electromagnetic energy is finite (zero) despite the $1/r^6$ singularity in the functional form of the density θ^{00}, and its self-force is zero. We have not had to speculate about the form of the dipole's material energy tensor other than to assume that it satisfied $\partial\cdot K=0$ in the static but interacting case, as it would if it were noninteracting. Presumably, however, in the time-dependent case the form of K will have to be specified quite closely in order to force θ to require a unique definition.

We have been considering the dipole from a frame of reference in which it is at rest. The components of the energy tensor we have calculated can be used, by following the rules of the Lorentz transformation, to find the components in any other frame in uniform motion with respect to the rest frame. But it is probably simpler to work with the geometrical quantities in space-time straightaway, rather than their components. Using the notation of Rowe (1978) we can verify that the four-dimensional relativistic energy tensor, off the world line, is the natural generalization of (17):

$$\theta = -\frac{1}{64\pi}(\partial^2)^2\left\{\frac{RR}{\rho^6}\left[(p\cdot R)^2 - p^2\rho^2\right]\right\} \qquad (\rho \neq 0) \tag{20}$$

The world line is straight in space-time and parallel to a constant timelike velocity vector v; the constant dipole moment vector p is perpendicular to v.

Equation (20) gives, when decomposed along the axes of the rest frame, precisely (16) and (17). It appears to give something nonzero for θ^{0s}, but does not in fact since

$$(\nabla^2)^2\left\{\frac{\mathbf{x}}{r^5}\left[(\mathbf{p}\cdot\mathbf{x})^2 - \mathbf{p}^2 r^2\right]\right\} = \mathbf{0} \qquad (r > 0)$$

Nonetheless, when the right-hand side of (20) is generalized to its distribution theory form, by the appropriate interpretation of the derivatives, one must add to it the expression

$$-\frac{1}{16}\int d\tau\left(\frac{2}{3}p^2\right)\left[(v,\partial) - gv\cdot\partial\right]\partial^2\delta(x-z) \tag{21}$$

concentrated on the world line, in order that $\partial\cdot\theta = 0$ should be satisfied. The additional term gives a nonzero contribution in the rest frame for the θ^{0s} terms only.

The further generalization of (20), to the case of an arbitrarily moving, time-dependent dipole, involves considerable calculation. It has been done by one of us in collaboration with G. T. Rowe and will be published separately.

3. QUADRUPOLES AND HIGHER MULTIPOLES

The generalization of the theory of the previous section to multipoles of higher order than the dipole requires nothing extra but a tractable method of handling the vector analysis. In this section only we use a special notation designed to distinguish clearly between dipoles, quadrupoles, octupoles, etc. We suppress the spatial point $\mathbf{x}$ and show explicitly the vectors that determine the multipole.

For example, in terms of the dipole potential and electric field, now written

$$\phi(\mathbf{p}) = (-\mathbf{p}\cdot\nabla)\frac{1}{r} \tag{22}$$

$$\mathbf{E}(\mathbf{p}) = \nabla(\mathbf{p}\cdot\nabla)\frac{1}{r} \tag{23}$$

we can write the quadrupole potential as

$$\phi(\mathbf{q},\mathbf{p}) = (-\mathbf{q}\cdot\nabla)(-\mathbf{p}\cdot\nabla)\frac{1}{r} = (-\mathbf{q}\cdot\nabla)\phi(\mathbf{p}) = \mathbf{q}\cdot\mathbf{E}(\mathbf{p}) \tag{24}$$

The quadrupole electric field can be expressed in terms of the octupole potential

$$\mathbf{E}(\mathbf{q},\mathbf{p}) = -\nabla\phi(\mathbf{q},\mathbf{p}) = \frac{\partial}{\partial\mathbf{s}}\phi(\mathbf{s},\mathbf{q},\mathbf{p}) \tag{25}$$

From the general definition of the momentum flux dyadic θ^{ss} and the energy density θ^{00} we have, in the region $r>0$, for the dipole case,

$$\mathbf{E}(\mathbf{p})\mathbf{E}(\mathbf{p}) = -4\pi\theta^{ss}(\mathbf{p}) + 4\pi I\theta^{00}(\mathbf{p}) \qquad (r>0) \tag{26}$$

Taking the scalar product left and right with $\mathbf{q}$, and using (24) with (16) and (17), then gives a formula for the quadrupole case with $[\phi(\mathbf{q},\mathbf{p})]^2$ expressed as a differential operator acting on an integrable function:

$$\begin{aligned}[\phi(\mathbf{q},\mathbf{p})]^2 &= [\mathbf{q}\cdot\mathbf{E}(\mathbf{p})]^2 \\ &= \frac{1}{16}(\nabla^2)^2\left\{\frac{\left[(\mathbf{q}\cdot\mathbf{x})^2-\mathbf{q}^2r^2\right]\left[(\mathbf{p}\cdot\mathbf{x})^2-\mathbf{p}^2r^2\right]}{r^6}\right\} \qquad (r>0)\end{aligned} \tag{27}$$

The energy density for the quadrupole may be expressed in a similar form by using (27) and a simple consequence of the Leibniz formula which is valid for any point multipole at $\mathbf{0}$:

$$\mathbf{E}^2 = \nabla\phi\cdot\nabla\phi = \tfrac{1}{2}\nabla^2(\phi^2) \qquad (r>0) \tag{28}$$

In this way we get

$$\theta^{00}(\mathbf{q},\mathbf{p}) = \frac{1}{256\pi}(\nabla^2)^3\left\{\frac{\left[(\mathbf{q}\cdot\mathbf{x})^2 - \mathbf{q}^2 r^2\right]\left[(\mathbf{p}\cdot\mathbf{x})^2 - \mathbf{p}^2 r^2\right]}{r^6}\right\} \qquad (r>0) \tag{29}$$

The structure of (29), when compared with (16) and (17), suggests a form for $\theta^{ss}(\mathbf{q},\mathbf{p})$ (and indeed a generalization to higher multipoles) which is not in fact valid. Instead, for the quadrupole flux tensor there is a rather complicated formula which was deduced, essentially by trial and error, using the explicit expression for $\mathbf{E}(\mathbf{q},\mathbf{p})$:

$$\begin{aligned}\theta^{ss}(\mathbf{q},\mathbf{p}) = \frac{1}{256\pi}(\nabla^2)^3\Bigg\{&\frac{\mathbf{xx}(\mathbf{q}\cdot\mathbf{x})^2(\mathbf{p}\cdot\mathbf{x})^2}{r^8}\\ &-\frac{4}{3}\frac{\mathbf{xxq}\cdot\mathbf{pq}\cdot\mathbf{xp}\cdot\mathbf{x}}{r^6} - \frac{1}{3}\frac{\mathbf{xx}\left[\mathbf{q}^2(\mathbf{p}\cdot\mathbf{x})^2 + \mathbf{p}^2(\mathbf{q}\cdot\mathbf{x})^2\right]}{r^6}\\ &+\frac{1}{3}\frac{\mathbf{xxq}^2\mathbf{p}^2}{r^4} + \frac{2}{3}\frac{\mathbf{xx}(\mathbf{q}\cdot\mathbf{p})^2}{r^4}\Bigg\}\\ -\frac{1}{192\pi}(\nabla^2)^2\Bigg\{&\frac{1}{2}(\mathbf{q}\cdot\nabla)^2\frac{\mathbf{xx}(\mathbf{p}\cdot\mathbf{x})^2}{r^6} + \frac{1}{2}(\mathbf{p}\cdot\nabla)^2\frac{\mathbf{xx}(\mathbf{q}\cdot\mathbf{x})^2}{r^6}\\ &-\mathbf{q}\cdot\nabla\mathbf{p}\cdot\nabla\frac{\mathbf{xxq}\cdot\mathbf{xp}\cdot\mathbf{x}}{r^6} + \mathbf{q}\cdot\nabla\mathbf{p}\cdot\nabla\frac{\mathbf{xxq}\cdot\mathbf{p}}{r^4}\\ &-\frac{1}{2}(\mathbf{p}\cdot\nabla)^2\frac{\mathbf{xxq}^2}{r^4} - \frac{1}{2}(\mathbf{q}\cdot\nabla)^2\frac{\mathbf{xxp}^2}{r^4}\Bigg\} \qquad (r>0)\end{aligned} \tag{30}$$

The heavy calculation involved in searching for (30) was much reduced by using a simplified version of a FORTRAN program prepared (G. T. Rowe, 1980) for treating the problem mentioned at the end of Section 2: the rigorous definition of the relativistic energy tensor for an arbitrarily accelerating and precessing point dipole. The program is a collection of subroutines which act on a set of terms each of the form

$$[\text{coefficient}] \quad [\text{multiadic}] \quad [\text{pairs}] \quad [\text{exponent of } 1/r]$$

The multiadic stands for a direct product of vectors **x**, **p**, **q**, etc., and the pairs are those in the scalar products of vectors. Subroutines for the gradient (which increases the multiadic index by one) and for various traces (which reduce the index by two) are easily constructed. The most important subroutine is the one which collects terms after they have been put in a canonical form.

We have not found an elegant generalization of the formulas (29) and (30), but a definite algorithm can be constructed to express θ^{ss} and θ^{00} as derivatives of integrable functions. We illustrate the procedure with the quadrupole. In this discussion all formulas are understood to be valid only in the region $r > 0$.

It is relatively easy to express ϕ^2 for a 2^n-pole as $(\nabla^2)^n$ acting on an integrable function—we will sketch how to do this in a moment. When this is done for an octupole, we can deduce **EE** for a quadrupole from the relation

$$\begin{aligned}[\phi(\mathbf{s},\mathbf{q},\mathbf{p})]^2 &= \left[\mathbf{s}\cdot\nabla\,\mathbf{q}\cdot\nabla\,\mathbf{p}\cdot\nabla\frac{1}{r}\right]^2 \\ &= \mathbf{s}\cdot[\mathbf{E}(\mathbf{q},\mathbf{p})\mathbf{E}(\mathbf{q},\mathbf{p})]\cdot\mathbf{s} \end{aligned} \tag{31}$$

From the trace of **EE** we get $\theta^{00}(\mathbf{q},\mathbf{p}) = \mathbf{E}^2/8\pi$ and then

$$\theta^{ss}(\mathbf{q},\mathbf{p}) = -\frac{1}{4\pi}\left[\mathbf{E}(\mathbf{q},\mathbf{p})\mathbf{E}(\mathbf{q},\mathbf{p}) - \frac{I}{2}\mathbf{E}^2(\mathbf{q},\mathbf{p})\right] \tag{32}$$

This is, of course, just the opposite of the sort of procedure that led from (26) to (27).

In order to calculate ϕ^2 as a power of ∇^2 times an integrable function, we first calculate ϕ^2 explicitly in the form

$$\phi^2 = \sum_{l,L}\frac{P_L(\mathbf{x})}{r^l} \tag{33}$$

where the $P_L(\mathbf{x})$ are homogeneous polynomials of degree L. In the case of the octupole we have

$$\phi^2 = \frac{9}{r^{14}}[r^2(\mathbf{p}\cdot\mathbf{s}\,\mathbf{q}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{s}\,\mathbf{p}\cdot\mathbf{x} + \mathbf{p}\cdot\mathbf{q}\,\mathbf{s}\cdot\mathbf{x}) - 5\mathbf{p}\cdot\mathbf{x}\,\mathbf{q}\cdot\mathbf{x}\,\mathbf{s}\cdot\mathbf{x}]^2 \tag{34}$$

Both l and L are even, and for a 2^n-pole, $l - L = 2(n+1)$.

To convert (33) to the required form, we use the formula

$$\nabla^2 \frac{P_L(\mathbf{x})}{r^{l-2}} = \frac{\nabla^2 P_L(\mathbf{x})}{r^{l-2}} + \frac{(l-2)[l-3-2L]P_L(\mathbf{x})}{r^l} \tag{35}$$

n times on each term. If the trouble has been taken to arrange that the P_L are harmonic, that is, $\nabla^2 P_L = 0$, then (35) can be applied n times to each term independently in (33). But if the P_L are not harmonic one must start with the largest value of L and work towards the smaller, redefining the lower polynomials at each stage to take account of the contribution of the terms of the form $\nabla^2 P_L / r^{l-2}$.

The result of this procedure, for the octupole, is

$$\begin{aligned}
[\phi(\mathbf{s},\mathbf{q},\mathbf{p})]^2 = \frac{(\nabla^2)^3}{64} \Bigg\{ & -\frac{(\mathbf{p}\cdot\mathbf{x})^2(\mathbf{q}\cdot\mathbf{x})^2(\mathbf{s}\cdot\mathbf{x})^2}{r^8} \\
& + \frac{(\mathbf{p}\cdot\mathbf{x})^2(\mathbf{q}\cdot\mathbf{x})^2\mathbf{s}^2 + (\mathbf{p}\cdot\mathbf{x})^2\mathbf{q}^2(\mathbf{s}\cdot\mathbf{x})^2 + \mathbf{p}^2(\mathbf{q}\cdot\mathbf{x})^2(\mathbf{s}\cdot\mathbf{x})^2}{r^6} \\
& - \frac{1}{3}\frac{(\mathbf{p}\cdot\mathbf{x})^2(\mathbf{q}\cdot\mathbf{s})^2 + (\mathbf{q}\cdot\mathbf{x})^2(\mathbf{p}\cdot\mathbf{s})^2 + (\mathbf{s}\cdot\mathbf{x})^2(\mathbf{p}\cdot\mathbf{q})^2}{r^4} \\
& + \frac{2}{3}\frac{\mathbf{p}\cdot\mathbf{x}\,\mathbf{q}\cdot\mathbf{x}\,\mathbf{p}\cdot\mathbf{s}\,\mathbf{q}\cdot\mathbf{s} + \mathbf{q}\cdot\mathbf{x}\,\mathbf{s}\cdot\mathbf{x}\,\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}\cdot\mathbf{s} + \mathbf{p}\cdot\mathbf{x}\,\mathbf{s}\cdot\mathbf{x}\,\mathbf{p}\cdot\mathbf{q}\,\mathbf{q}\cdot\mathbf{s}}{r^4} \\
& - \frac{2}{3}\frac{(\mathbf{p}\cdot\mathbf{x})^2\mathbf{q}^2\mathbf{s}^2 + (\mathbf{q}\cdot\mathbf{x})^2\mathbf{p}^2\mathbf{s}^2 + (\mathbf{s}\cdot\mathbf{x})^2\mathbf{p}^2\mathbf{q}^2}{r^4} \\
& - \frac{2}{3}\frac{\mathbf{q}\cdot\mathbf{x}\,\mathbf{s}\cdot\mathbf{x}\,\mathbf{p}^2\mathbf{q}\cdot\mathbf{s} + \mathbf{p}\cdot\mathbf{x}\,\mathbf{s}\cdot\mathbf{x}\,\mathbf{q}^2\mathbf{p}\cdot\mathbf{s} + \mathbf{q}\cdot\mathbf{x}\,\mathbf{p}\cdot\mathbf{x}\,\mathbf{s}^2\mathbf{q}\cdot\mathbf{p}}{r^4} \\
& + \frac{2}{3}\frac{\mathbf{p}^2\mathbf{q}^2\mathbf{s}^2 - \mathbf{p}\cdot\mathbf{q}\,\mathbf{p}\cdot\mathbf{s}\,\mathbf{q}\cdot\mathbf{s}}{r^2} \\
& + \frac{1}{3}\frac{(\mathbf{p}\cdot\mathbf{q})^2\mathbf{s}^2 + (\mathbf{q}\cdot\mathbf{s})^2\mathbf{p}^2 + (\mathbf{p}\cdot\mathbf{s})^2\mathbf{q}^2}{r^2} \Bigg\}
\end{aligned} \tag{36}$$

Taking (36) as the left-hand side in (31), we deduce $\theta^{ss}(\mathbf{q},\mathbf{p})$ from the form (32):

$$\theta^{ss}(\mathbf{q},\mathbf{p}) = \frac{1}{256\pi}(\nabla^2)^3\left\{\frac{\mathbf{xx}(\mathbf{p}\cdot\mathbf{x})^2(\mathbf{q}\cdot\mathbf{x})^2}{r^8} - \frac{\mathbf{xx}\left[\mathbf{p}^2(\mathbf{q}\cdot\mathbf{x})^2+\mathbf{q}^2(\mathbf{p}\cdot\mathbf{x})^2\right]}{r^6}\right.$$
$$+\frac{\mathbf{pp}(\mathbf{q}\cdot\mathbf{x})^2+\mathbf{qq}(\mathbf{p}\cdot\mathbf{x})^2}{3r^4}$$
$$+\frac{\mathbf{xx}\left[(\mathbf{p}\cdot\mathbf{q})^2+2\mathbf{p}^2\mathbf{q}^2\right]}{3r^4}$$
$$-\frac{(\mathbf{p},\mathbf{q})\mathbf{p}\cdot\mathbf{x}\,\mathbf{q}\cdot\mathbf{x}+(\mathbf{p},\mathbf{x})\mathbf{p}\cdot\mathbf{q}\,\mathbf{q}\cdot\mathbf{x}+(\mathbf{q},\mathbf{x})\mathbf{p}\cdot\mathbf{q}\,\mathbf{p}\cdot\mathbf{x}}{3r^4}$$
$$+\frac{(\mathbf{q},\mathbf{x})\mathbf{q}\cdot\mathbf{x}\,\mathbf{p}^2+(\mathbf{p},\mathbf{x})\mathbf{p}\cdot\mathbf{x}\,\mathbf{q}^2}{3r^4}$$
$$-\frac{I\left[(\mathbf{p}\cdot\mathbf{x})^2\mathbf{q}^2+(\mathbf{q}\cdot\mathbf{x})^2\mathbf{p}^2-2\mathbf{p}\cdot\mathbf{x}\,\mathbf{q}\cdot\mathbf{x}\,\mathbf{p}\cdot\mathbf{q}\right]}{3r^4}$$
$$\left.+\frac{I\left[\mathbf{p}^2\mathbf{q}^2-(\mathbf{p}\cdot\mathbf{q})^2\right]}{3r^2} - \frac{\mathbf{pp}\mathbf{q}^2+\mathbf{qq}\mathbf{p}^2-(\mathbf{p},\mathbf{q})\mathbf{p}\cdot\mathbf{q}}{3r^2}\right\} \tag{37}$$

where the notation $(\mathbf{p},\mathbf{q})$ means $\mathbf{pq}+\mathbf{qp}$.

Equation (37) is an alternative derivative form for $\theta^{ss}(\mathbf{q},\mathbf{p})$; either (37) or the earlier (30) can be used as a distribution theory definition which will ensure that the force density $\mathbf{f} = -\nabla\cdot\theta^{ss}$ is zero for this model of the static quadrupole. The argument is based simply on $\nabla\cdot\theta^{ss}=0$ for $r>0$ and

$$\int_{r=\epsilon} d\Omega[\text{odd number of factors of } \mathbf{x}] = \mathbf{0}$$

The argument leading from (29) to zero total energy is also simple and short —it is based on the derivative form and is the same as for the point charge or the dipole.

The arguments given here can be generalized to any higher point multipoles. A consistent static theory can be achieved for any of them. Much more difficult is the task of getting a dynamic theory for an arbitrarily moving point multipole in which energy-momentum is genuinely conserved.

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